Hedging Volatility Risk

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Abstract

Volatility derivatives are becoming increasingly popular as means for hedging unexpected changes in volatility. Although pricing volatility derivatives demands extreme care in modeling the underlying volatility process, not much attention has been devoted to the complete specification of the autonomous process that volatility follows in continuous time. Despite the fact that jumps are widely considered as a salient feature of volatility, their implications for pricing and hedging volatility options and futures are not yet fully understood. This thesis addresses two principal issues: a) modeling volatility, and b) hedging volatility risk.

With respect to the modeling of volatility, the thesis assesses the ability of the most commonly used processes (diffusion and jump-diffusion) to capture the dynamics of implied volatility. The assessment of these processes is performed both analytically and empirically using data from the implied volatility index VIX for a period of 15 years. The empirical analysis produces new evidence concerning stationarity, long-memory, non-normality and jump behavior. Furthermore, the empirical fit of diffusion processes can be significantly improved by the addition of jumps. If jumps are conditioned on the level of the index, model performance is further enhanced.

With respect to hedging volatility risk, the thesis assesses the hedging effectiveness of volatility derivatives. The comparative evaluation attests that the most “naïve” volatility option pricing model can be reliably used for pricing and hedging purposes. Finally, the thesis develops new closed-form models for pricing volatility futures and European volatility options, when the underlying volatility index displays mean reversion and jumps. The results from these new models demonstrate that incorrectly omitting jumps may cause severe problems to both pricing and hedging.

1. Introduction

Volatility is undoubtedly the most important variable in finance. It appears consistently across a wide spectrum of theories and applications in asset pricing, portfolio theory, risk management, derivatives, corporate finance, investment valuation and financial econometrics. It is widely recognized that volatility changes stochastically over time. This makes volatility a latent variable i.e. it cannot be directly observed. A plethora of alternative measures and approaches have been developed in academic research and industry in order to measure volatility. Volatility risk (vega risk) is the exposure to the changes in volatility.

This thesis deals with hedging volatility risk. Hedging volatility risk is important to investors ranging from individuals to financial institutions and pension funds. An often cited example of the importance of volatility risk concerns the remarkable shift in volatility that followed the 1987 crash. Bad estimate and/or ineffective hedging of volatility risk (along with excessive leverage and fraud) resulted to the collapse of major financial institutions in the recent past e.g. Barings Bank, Long Term Capital Management. In hedging volatility risk we have to address three issues. First, robust volatility measure has to be specified. Second, the properties and the dynamics of this particular measure have to be modeled. Finally, the suitable hedging instrument has to be priced.

Traditionally, the hedging of volatility risk has been carried out using exchange-traded standard futures and plain-vanilla options. However, these instruments are designed so as to deal with price risk, primarily. A recent development has been the treatment of volatility as a distinct asset which can be packaged in an index (volatility index), and traded using volatility derivatives (swaps, futures and options). Volatility derivatives are instruments whose underlying asset is some measure of volatility. They are considered by some to “have the potential to be one of the most important new financial innovations” (Grubichler and Longstaff, 1996). Volatility derivatives are natural candidates to hedge volatility risk.

The first volatility index, named VIX (currently termed VXO), was introduced in 1993 by the Chicago Board Options Exchange (CBOE). It was estimated on the basis of implied volatilities from at-the-money options on the SP100 index, using a methodology proposed by Whaley (1993). In 2003, the CBOE adopted a new methodology to calculate VIX as an average of out-of-money option prices across all available strikes on the S&P 500 index. Ever since, several other implied volatility indices have been developed, including: the VXN and VXD in the CBOE, the VDAX in Germany, the VX1 and VX6 in France, the VSTOXX in the Eurex, the VSMI in Switzerland, the MVX in Canada, etc.
Volatility indices are used in a number of applications. They serve as the underlying asset to volatility derivatives; they could play the same role as the market index plays for options and futures on the index. Volatility indices express market expectations. The financial press, e.g. CNBC, Barrons, Wall Street Journal, regularly refers to the VIX volatility index as the "investor fear gauge". Regulatory bodies and central banks, such as the Bank of England, have used the VIX to depict equity uncertainty and relate it to subsequent movements in other variables, such as swap spreads. Fleming et al. (1995), Moraux et al. (1999), Blair et al. (2001), Corrado and Miller (2003), Simon (2003), Giot, (2005b), have demonstrated the practical importance of volatility indices as an efficient, yet biased, forecast of future market volatility. Volatility indices can also be used for Value-at-Risk purposes (Giot, 2005a), and to identify buying/selling opportunities in the stock market (Whaley, 2000). Despite the growing number of studies on the importance of volatility indices to both academics and practitioners, few are concerned with the properties of these indices in continuous time. Moreover, the process that best describes the dynamics of volatility indices is still an open issue.

Options and futures written on a volatility index were first suggested by Brenner and Galai (1989, 1993) as a response to the growing need for the creation of instruments to hedge volatility risk. Grubichler and Longstaff (1996) developed the first models for the valuation of futures and European-style options written on instantaneous volatility. The authors' assumptions is that the underlying volatility follows a mean reverting square root process, similar to that used earlier by Heston (1993). Determined and Osakwe (2000) provides analytical formulas to price both American and European-style volatility options assuming a mean-reverting log volatility model. Heston and Nandi (2000a) derive analytical solutions in both discrete and continuous time for pricing European options written on variance. These were based on a discrete-time GARCH volatility process and its continuous time counterpart developed by Heston and Nandi (2000b). Recently, Daouk and Guo (2004) studied the valuation of volatility options based on a Switching Regime Asymmetric GARCH process for the underlying. All the above models, which are based on a diffusion process, do not account for jumps in the underlying volatility process, although jumps are widely considered as a salient feature of volatility (see among others, Duffie et al., 2000, Bakshi and Cao, 2004, Broadie et al., 2004, Eraker, 2004).

Volatility derivatives have a wide range of important applications for all market participants. It has been argued that volatility derivatives make the markets more complete since they expand the available set of investment opportunities and allow direct hedging of volatility risk, without necessarily resorting to dynamical adjustments. Investment funds employ volatility derivatives for insuring against movements in volatility (vega hedging). Certain classes of investors, such as convertible bond arbitrage funds and structured product issuers, can use these derivatives to insure against their structural exposure to volatility. Investors can employ them to partially insure against shifts in transaction costs and tracking error penalties, both of which increase during periods of high uncertainty. Investment managers may use these derivatives to hedge against the risks of a so-called high-correlation environment. This is because asset correlations have been found to increase significantly during periods of high volatility, making active asset picking and portfolio diversification very difficult (see for further discussion Ang and Chen, 2002, Skintzi and Refenes, 2005). As volatility is a key input for risk management and capital adequacy models, such as the VaR, volatility derivatives could be used by banks as a shield against shifts in volatility and correlation during stress market conditions. Since shifts in equity risk have a significant impact on risk premia, firms could employ volatility derivatives to protect themselves from unexpected changes in cost of capital. Although not available yet, bond and foreign exchange volatility indices and derivatives, would allow firms that are exposed to volatility in these markets to hedge against changes in volatility. Finally, ample liquidity in this market is provided by traders since volatility derivatives can provide the most efficient and low-cost way for speculating against changes in volatility (for a review of the volatility trading/hedging techniques see Carr and Madan, 1998).

Motivated by the developments in the industry and academia stock exchanges around the world have introduced (or plans to introduce) volatility derivatives. The CBOE introduced in March 2004 volatility futures on the implied volatility measured by the VIX index. The CBOE recently also announced the imminent introduction of volatility futures on the implied volatility index VXD along with volatility options.2 Eurex has

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2 The previous attempt to introduce contracts on volatility in an organized market was undertaken by the German Exchange in 1997; that was a volatility future (VOLAX) on the German implied volatility index VDAX. However, the trading of VOLAX ceased in 1998. An anecdotal explanation that is offered by practitioners for the failure of VOLAX, as well as for the delay in introducing volatility options, is that market makers are neither familiar
launched in September 2005 three new volatility futures on the VDAX-NEW, VSTOXX and VSMI volatility indices, respectively.

However, the hedging effectiveness of volatility derivatives compared to that of plain-vanilla derivatives has not yet been studied. Jiang and Oomen (2001) have examined the hedging performance only of volatility futures versus standard options. This is surprising given that one of the main arguments for introducing volatility derivatives is based on their use as hedging instruments. Furthermore, the comparative hedging and pricing performance of the existing volatility derivatives pricing models in the presence of model error has attracted very little attention. Daouk and Guo (2004) have focused on the pricing side and they have investigated the impact of model error to the performance of only one (Gümbichler and Longstaff 1996) of the developed volatility option pricing models. Still, the question “what is the impact of using a mis-specified process on the hedging and pricing performance of the volatility derivatives pricing models?” remains unanswered.

Pricing volatility derivatives demands extreme care in modeling the underlying volatility process. There are good reasons to believe that mispecifying the volatility process when the option is written on a stock index is much less serious than in the case where the option is written on a volatility index. In the first case, the volatility process is itself the underlying asset. While the dynamics of the instantaneous volatility have been considered extensively, this is not the case for those of implied volatility.

The literature on the specification of the stochastic process that governs the dynamics of instantaneous volatility in continuous time first

with the models that have been developed to price volatility futures and options, nor with their use for hedging purposes. In accordance with this claim, Whaley (1998) also states “In summary, I believe that volatility derivatives are a viable exchange-traded product… I also believe that the contracts have not been successful largely because potential market makers have not stepped forward. The reason is fear.”

3 The implied volatility is usually used as a proxy for the instantaneous volatility. Usually, it is interpreted as the average instantaneous volatility to be realized over the life of the option. However, this is strictly true in a Hull and White (1987) world, where the instantaneous volatility is uncorrelated with the asset price, the market price of volatility risk is zero, and the option is linear with respect to volatility (i.e. at-the-money). In the case where these conditions do not hold, the implied differs from the instantaneous volatility.

emerged in a stochastic volatility option-pricing context. In this setup, the underlying asset price and the instantaneous volatility of the underlying asset returns are modeled jointly. In the late eighties, many stochastic volatility option pricing models were developed by assuming a process with continuous paths that drives the volatility of the underlying asset price returns (see, among others, Hull and White, 1987, Johnson and Shanno, 1987, Scott, 1987, Wiggins, 1987, Stein and Stein, 1991, Heston, 1993, and Jones, 2003 for a more flexible specification); the underlying asset price was also assumed to follow a diffusion process. In the late nineties, new type stochastic volatility option pricing models were introduced. These models were based on a jump diffusion process for the underlying asset price and on a diffusion volatility process (see e.g., Bakshi et al, 1997, Bates, 1996, 2000, Andersen et al., 2002, and Pan, 2002). Recently, there have appeared option-pricing models that are founded on “double jump” processes: both the underlying asset price and the instantaneous volatility follow jump-diffusion processes (see e.g., Duffie et al., 2000, Bakshi and Cao, 2004, Broadie et al., 2004, Eraker, 2004). These models have been shown to be superior to models without jumps in volatility in terms of fitting traded index and equity options price series (see e.g., Eraker et al., 2003, and in a Value-at-Risk framework Lehar et al., 2002).

However, not much attention has been devoted to the complete specification of the autonomous process that implied volatility follows in continuous time 4. Kamal and Derman (1997), Skiadopoulos et al. (1999), Alexander (1998), Ané and Labidi (2001), Cont and da Fonseca (2002), and Fengler et al. (2003) have studied only the volatility structure of diffusion implied volatility processes, i.e. the number and form of shocks that drive implied volatilities over time. These studies have left unanswered the specification and estimation of the drift, though. Merville and Piepeta (1989), Moraux et al. (1999), and Daouk and Guo (2004) estimated diffusion implied volatility mean-reverting processes. However, no comparison with alternative processes was performed, and no jumps in volatility were considered. Recently, Bakshi et al. (2006) estimated various general specifications of the autonomous instantaneous variance diffusion process (i.e. they assume non-linear mean-reversion). To this end, they used the squared

4 In a discrete-time context, Poterba and Summers (1986) were among the first to document that implied volatility mean-reverts. The paper by Franks and Schwartz (1991) suggests that the changes in implied volatility can be regarded as being stochastic, since they are attributed to shocks in various economic variables such as inflation and nominal interest rates.
implied volatility index VIX as a proxy for the unobserved instantaneous variance. However, the empirically documented presence of jumps in implied volatility (see e.g., Malz, 2000, Wagner and Szimayer, 2004) was not addressed in their study. Finally, Wagner and Szimayer (2004) were the first to investigate the presence of jumps in implied volatility by estimating an autonomous mean reverting jump diffusion process using data on the implied volatility indices VIX and VDAX. They found evidence of significant positive jumps in implied volatilities. However, they adopted the rather restrictive assumption that the volatility jump size is constant rather than being random. Again, their specification was not compared with alternative ones.

The objective of this thesis is to fill in the gaps concerning: a) modeling volatility and b) hedging volatility risk. With respect to the modeling of volatility, the thesis answers the following question: “Which stochastic volatility process should we use?” Towards this direction, first it taxonomizes many of the more popular diffusion processes, currently used by academic researchers and practitioners to model the dynamics of volatility, in continuous time setting. The processes are nested in a general stochastic differential equation, and their properties are analyzed in a unified manner. This common framework facilitates the evaluation of their relative performance in a consistent way.

Second, the thesis empirically assesses the ability of the diffusion processes to capture the dynamics of the autonomous volatility over time, using daily data on the VIX index for a period of 15 years. The empirical analysis produces new evidence concerning long-memory, nonlinearities and jump behavior in VIX. The models that perform best in capturing the dynamics of implied volatility are those that incorporate mean reversion, and allow volatility of volatility to be highly sensitive to the current level of volatility. However, none of the diffusion models can generate nonnormalities consistent with those observed in the data.

Third, the thesis estimates and compares diffusion and jump-diffusion stochastic volatility processes using both the Generalized Method of Moments (GMM) and the Maximum Likelihood Estimation (MLE). The estimation results demonstrate that the empirical fit of diffusion processes can be significantly improved by the addition of jumps. Jumps play a dominant role in implied volatility dynamics especially during periods of market stress. If jumps are conditioned on the level of the index, model performance is further enhanced. It is also demonstrated that this process can produce unconditional distributions that closely resemble that of the actual data. These findings are consistent with the research on the joint dynamics of volatility and asset returns which has also suggested the presence of jumps in volatility (e.g., Duffie et al, 2000; Eraker et al, 2003; Eraker, 2004).

With respect to hedging volatility risk, this thesis makes three contributions. First, it assesses the hedging effectiveness of volatility derivatives. It compares the performance of a hedge using volatility options versus a hedge using standard European options, and it evaluates the comparative pricing and hedging performance of various volatility option pricing models in the presence of model error. Alternative dynamic hedging schemes are compared, and various option-pricing models are considered. It comes to light that volatility options are not better hedging instruments than plain-vanilla options. Furthermore, the comparative evaluation attests that the most “naïve” volatility option pricing model can be reliably used for pricing and hedging purposes.

Second, the thesis develops closed form expressions for pricing futures and European options on volatility, assuming a mean reverting square root process with jumps. The proposed option pricing model nests, as a special case, the model by Grunbichler and Longstaff (1996). It is based on the same volatility dynamics as those implied by the most promising option pricing models founded on “double jump” processes. As mentioned earlier in this Section, one of the main reasons for the introduction of volatility derivatives is to increase the set of hedging instruments available to investors. Thus, it makes sense from a risk management perspective, to use the same model for the autonomous volatility process, and the joint dynamics of volatility and asset returns.

Finally, the thesis assesses the implications of incorrectly omitting jumps from the volatility process for pricing and hedging volatility derivatives. It is demonstrated that prices and hedge ratios may differ substantially. The model without jumps in volatility (i.e., the Longstaff and Grunbichler, 1996, model) significantly undervalues (overvalues) short (long) maturity volatility options. Moreover, it is more sensitive to changes in the underlying volatility.

The thesis is organized as following: Section 2 analytically compares the various stochastic volatility diffusion processes; derives the advantages and the disadvantages of each process and demonstrates the implications and the relationships between them as far as the option pricing, risk management, and asset allocation are concerned. Section 3 analyses the empirical behavior of the VIX for a period of 15 years. Next, it describes the econometric approach, and finally discusses the empirical results from comparing the stochastic volatility diffusion processes considered in Section 2. Section 4 presents and discusses the
results on the hedging performance and the robustness of the volatility option pricing models for hedging and pricing purposes in the presence of model error. Section 5 analyzes the mean-reverting square root volatility process augmented with jumps; describes the econometric approach and presents the empirical results from comparing the jump-diffusion processes with the diffusion processes considered in Section 3. Next, it develops the valuation formulae for volatility derivatives when the underlying volatility follows a mean-reverting jump-diffusion process. Finally, it presents the properties of these models and the importance of jumps from the perspective of pricing and risk management. The thesis terminates with Section 6, giving a summary, the major conclusions and suggestions for further research.

2. Stochastic Volatility: Properties and Implications for Option Pricing, Risk Management, and Asset Allocation

In this section, I summarized and compared the most popular continuous time stochastic processes that have been widely used to model the dynamics of instantaneous volatility. I clarified the mathematical properties, and I discussed the ability of each volatility process to capture the stylized facts of volatility. The mathematical complexity increases as the model attempts to incorporate more stylized facts about volatility. Certain implications of stochastic volatility for option pricing, risk management and asset allocation have been discussed.

Table 1 summarizes the solutions of the presented volatility processes, and lists their distributional properties. All the processes exhibit the mean-reversion property. Positivity is retained only by the MRSRP and the MRLP. The MRSRP has the smallest variation, while the MRLP exhibits the greatest variation, as expected. Inspection of the processes shows important differences both in the drift and in the volatility structure. The drifts of the processes 1, 2 and 3 are proportional to $V_t$, while the drift of process 4 is proportional to a function of $V_t$. Solving process 1 for volatility yields that volatility grows exponentially as a function of time. On the other hand, processes 2, 3, 4, describe volatility as a mean reverting process. The MRSRP adds the smallest variation to the evolution of volatility, as compared to the other three processes.

The GBMP is tractable but it does not conform to the mean-reverting nature of volatility, a property that is observed empirically. On the other hand, the MRGP generates mean reverting paths of volatility, but it allows volatility to take negative values. The MRSRP constrains volatility to taking positive values. However, under the MRSRP process, volatility is not distributed log-normally any longer. Instead, it follows the more complex Chi-Square distribution. This makes the process less tractable. Finally, the MRLP does not allow negative values, it is tractable, it adds significant variation to volatility, and it exhibits a stronger mean reverting pattern.

3. VIX Empirical Properties and Comparative Evaluation of Stochastic Volatility Diffusion Processes

This section examines two main issues. Firstly, it extends the empirical literature on volatility indices using daily data on the VIX index for a period of 15 years. It confirms previous findings of mean reversion and heteroskedasticity and provides new evidence concerning stationarity, long-memory, non-normality and jump behavior. This analysis is particularly useful in understanding the dynamics underlying the VIX and selecting an appropriate mathematical process to describe them.

Secondly, using the data from VIX it explores for the first time the ability of alternative univariate diffusion to capture the dynamics of implied volatility over time. Generalized Methods of Moments (GMM) is used to estimate the parameters of the various volatility processes. Standard statistical tests are used to compare the alternative processes. From the econometric perspective, my study is analogous to studies that have been conducted in the interest rate literature where the validity of alternative processes for the short-term interest rate has been investigated (see e.g., Chan et al., 1992).

In order to capture the stochastic behavior of implied volatility, I estimate various continuous stochastic volatility processes, nested in the following stochastic differential equation:

$$dV_t = \lambda(\mu - V_t)dt + \sigma V_t dZ$$

The above process is the well-known Constant Elasticity of Variance (CEV) process, which is considered by Ait-Sahalia (1999), Chacko and Viceira (2003), and Jones (2003). These dynamics imply that the conditional mean and the variance of the above process depend on the level of $V_t$, through the term $?$. The parameters $\lambda, \mu, \sigma$ can be defined as the speed of mean reversion, the long-run mean and the volatility of volatility, respectively. The stochastic differential equation in (1) nests a broad class of stochastic volatility processes. Each model can be obtained by placing the appropriate restrictions on the four parameters $\lambda, \mu, \sigma, \sigma$. In this section, I focus on six different specifications of the dynamics of implied volatility that have appeared in literature. The various specifications of the differential equation are described in Table 2.

In the analysis I include two more processes than those that I have analyze in Section 1. The GARCH diffusion process is appealing because it’s the continuous time limit of many GARCH-type processes, as shown by Nelson (1990). Similar
processes to 3/2 process has been proposed by Heston (1997) and Lewis (2000). Note that both GARCH-diffusion Process and 3/2 Process does not have closed form density function.

I begin by estimating the unrestricted model (CEV model, equation (1)) and the six restricted volatility process. Following Chan et al. (1992), I use two criteria in order to assess further the relative performance of the models. First I examine the $\chi^2$ measure, which provides a goodness-of-fit test for the model. A high value of this statistic means that the model is misspecified. Second, the $R^2$, which provides information with respect to the ability of each model to forecast the changes in volatility (see Chan et al. (1992) for the derivation of the $\chi^2$ and $R^2$ coefficients). Table 3 shows the GMM results for VIX. For each one of the processes the estimated parameters, the $\chi^2$ criterion, and the coefficient of determination $R^2$, respectively, are reported.

Table 3 brings forward a number of points with interesting implications. First, the $\chi^2$ measure for goodness of fit suggests that all models are misspecified at the 99% confidence level. None of the above process is capable of capturing the dynamics of implied volatility. The least misspecified processes are, in descending order of the $J$ value, the GARCH-diffusion, the 3/2 process, and the MRSRP, respectively. Second, the volatility of the process $s$ is highly sensitive to the level of $V_t$. The unconstrained estimate of $\nu$ is 1.27 and statistically significant. Third, the mean reversion, as implied by the unrestricted model, appears to be strong and significant.

Though all models are misspecified, the ranking of the processes can lead to some important conclusions. The ranking can well be explained by taking into consideration the previously mentioned facts concerning the diffusion component and the mean reversion parameter. The GARCH-diffusion and 3/2 process are the least misspecified because they combine mean reversion in the drift component and the diffusion component depends on the level of $V_t$. Moreover, the $\nu$ parameter is closed to one implied by the unrestricted model. Therefore, the processes captures the mean reversion observed in the data and can generate partially, through the volatility component, the observed excess skewness and kurtosis. The MRSRP model has the same mean reversion structure, but the volatility of the process depends on the square root of $V_t$ (less sensitive).

An interesting phenomenon appears in the ranking between the GBMP and the MRGP process, respectively. This can be consider as a raw evidence that for some cases the addition of mean reversion is more important than the specification of the volatility component. The GBMP that does not incorporate any mean reversion and the volatility is linear with respect to the level of $V_t$, while MRGP incorporates mean reversion but the volatility of the process does not depend from the current level of $V_t$. Judging from the ranking of the processes mean reversion is more important that the specification of the volatility component.

As far as the forecast power for volatility changes (see the coefficient of determination $R^2$) we can see that GBMP have no explanatory power with respect to the changes of volatility $V_t$. The rest of models have similar explanatory power ranging from 0.9% to 1.42%. Interestingly, all the models fail to forecast the changes in volatility.

4. Assessing the Hedging Effectiveness, Pricing, and Model Error of Volatility Options

This section makes two contributions to the rapidly evolving volatility options literature by addressing the following two questions: (a) Are volatility options superior to standard options in terms of hedging volatility risk?, and (b) Are the volatility option pricing models robust for hedging and pricing purposes in the presence of model risk?

To this end, a joint Monte Carlo simulation of the stock price and volatility in a stochastic volatility setup has been employed. First, two alternative dynamic delta-vega with discrete rebalancing hedging schemes were constructed to assess the hedging performance of plain vanilla options versus volatility options. A short standard European call is assumed to be the option to be hedged (target option). A standard European call and a European volatility call option are the alternative hedging instruments. Black and Scholes (1973), Heston (1993), Whaley (1993), Grünbichler and Longstaff (1996), and Detemple and Osakwe (2000) models have been used to hedge the standard and volatility options. Then, the robustness of the hedging and pricing effectiveness of the volatility option pricing models in the presence of model error was investigated. The Grünbichler-Longstaff and Heston models were assumed to be the “true” models; this is consistent with the choice of the volatility data generating process used to simulate the data. My two research questions have been examined for various expiry dates and strikes of the target option, as well as for alternative correlation values between the stock price and volatility, and for different rebalancing frequencies. Roll-over and no-roll over strategies in the volatility option were also considered.

In terms of the hedging effectiveness, I found that the hedging scheme that uses volatility options as hedging instruments is not superior to the one that uses standard options. In terms of the impact of model error to the hedging performance,
combinations that use simpler models such as Black-Scholes and Whaley’s seem to perform equally well with the benchmark combination (Grünbichler-Longstaff and Heston). Regarding the impact of model error to the pricing performance, Whaley’s model performs better than Detemple-Osakwe’s model.

This section has at least five important implications for the use of volatility options and their pricing models. First, volatility options are not better hedging vehicles than plain-vanilla options for the purposes of hedging standard options. This finding extends the conclusions in Jiang and Oomen (2001) who examined the hedging effectiveness of volatility futures versus plain-vanilla options; they found that the latter perform better than the former. However, this does not invalidate the imminent introduction of volatility options (and volatility futures) in various exchanges. Volatility options may be proved to be very useful for volatility trading and for hedging other types of options, e.g., exotic options. The liquidity and the transaction costs will be critical factors for the success of this emerging new market, as well. Second, in the case that an investor chooses the volatility options as hedging instruments, these should be used to hedge at-the-money and out-of-the-money rather than in-the-money target options. This feature may encourage the use of volatility options for hedging purposes given that most of the options trading activity is concentrated on at-the-money options. Third, the hedging performance of volatility options increases as their rebalancing frequency increases. Fourth, the roll-over strategy with volatility options should be preferred since it performs better than the no-roll-over strategy. Finally, despite the fact that Whaley’s model is the worst misspecified model within my simulation framework, it can be reliably used to hedge standard options with volatility options, and to price volatility options. This is in accordance with the results from previous studies in the model error (standard options) literature that found that increasing the complexity of the option pricing model does not necessarily improve its pricing and hedging performance (see e.g., Bakshi et al., 1997; Dumas et al., 1998).

5. Pricing Volatility Futures and Options in the Presence of Jumps

Motivated by the growing literature on volatility derivatives and their imminent introduction in major exchanges, this section examines the empirical relevance and potential impact of volatility jumps in autonomous volatility option pricing and risk management.

Empirical analysis of the VIX over a period of 10 years, in Section 3, provided a wealth of evidence supporting the existence of some stationary, mean-reverting process with jumps. Motivated by the preliminary analysis, I concentrated on the popular mean-reverting square root process, originally proposed by Grünbichler and Longstaff (1996), and its augmentation by an upward jump. An ML estimation scheme was described and applied to the VIX data. The results suggested that the addition of jumps, especially if they are conditioned on the volatility level, improves significantly fitting ability. Moreover, simulation results suggested that the augmented model has the ability to produce non-normal distributions that closely resemble those of the original data. Closed form models for pricing futures and options were then developed assuming a square root mean reverting diffusion stochastic process that allows for positive jumps in volatility. The proposed volatility option pricing models appears to have comparable properties with existing models in the literature (Grünbichler and Longstaff 1996; Detemple and Osakwe, 2000). However, it was demonstrated that incorrectly omitting jumps in volatility may result in severe mispricing. In particular, in the case where there are upwards jumps in volatility, short (long) term volatility options are more expensive (cheaper) by about 25% (14%). In addition, volatility calls are far less sensitive to the changes of the underlying volatility by a factor of about two.

The findings in this section do not necessarily support criticism against the specific structural form assumed by existing volatility future and option pricing models. Rather, they attempt to demonstrate that pricing derivatives on a volatility index should carefully account for salient features of the data since the results obtained are particularly sensitive to the model used to approximate the underlying dynamics. Testing against actual market prices will provide more definitive evidence on the merit of alternative pricing models. In the case of futures this is possible since some data do exist for futures on volatility indices (a relevant empirical investigation is undertaken by Dotis et al., 2005). However, since no volatility options market data are yet available, I cannot fully test the empirical relevance of alternative option pricing models. However, it is crucial to fully understand the dynamics of the underlying and the implications of competing option pricing models in order to understand the peculiarities of this asset class and facilitate a smooth operation of the market when it operates.

6. Conclusions

In Section 2, I taxonomize many of the more popular continuous-time stochastic volatility diffusion processes currently used by academic researchers and practitioners to model the dynamics of volatility in continuous time setting. The
processes are nested in a general stochastic differential equation and their properties are analyzed in a unified manner. I consider four such processes: the Geometric Brownian Motion Process (GBMP), the Mean-Reverting Gaussian Process (MRGP), the Mean Reverting Square-Root Process (MRSRP), and the Mean-Reverting Logarithmic Process (MRLP). I clarified the mathematical properties, and I analytically compare the ability of each volatility process to capture the stylized facts of volatility. The mathematical complexity increases as the model attempts to incorporate more stylized facts about volatility. Certain implications of stochastic volatility for option pricing, risk management and asset allocation have been discussed.

The GBMP is tractable but it does not conform to the mean-reverting nature of volatility, a property that is observed empirically. On the other hand, the MRGP generates mean reverting paths of volatility, but it allows volatility to take negative values. The MRSRP constrains volatility to taking positive values. However, under the MRSRP process, volatility is not distributed log-normally any longer. Instead, it follows the more complex Chi-Square distribution. This makes the process less tractable. Finally, the MRLP does not allow negative values, it is tractable, it adds significant variation to volatility, and it exhibits a stronger mean reverting pattern.

In Section 3, I first concentrated on the empirical properties of the implied volatility index VIX. Empirical analysis of the VIX provided a wealth of evidence supporting the existence of some stationary, mean-reverting process and the existence of discontinuities (jumps). Second, I assessed the ability of the most commonly used stochastic volatility diffusion processes to capture the dynamics of implied volatility. The assessment of these processes is performed empirically using data from the implied volatility index VIX for a period of 15 years. In the analysis I included two more models than those considered in Section 2, that do not have closed form density functions: the GARCH diffusion process and the 3/2 process. All the models are nested within simple econometric framework that allows direct comparison. The processes are estimated using the Generalized Method of Moments. The estimation results show that one of the most important features of volatility evolution is the dependence of its volatility on the current level of volatility. The model that performs best among the competing models is the Constant Elasticity of Variance (CEV) model. However none of the models seem to be able to forecast the changes of volatility.

In Section 4 the following two questions were addressed: (a) Are volatility options superior to standard options in terms of hedging volatility risk?, and (b) Are the volatility option pricing models robust for hedging and pricing purposes in the presence of model risk? In terms of the hedging effectiveness, I found that the hedging scheme that uses volatility options as hedging instruments is not superior to the one that uses standard options. This finding extends the conclusions in Jiang and Oomen (2001) who examined the hedging effectiveness of volatility futures versus plain-vanilla options; they found that the latter perform better than the former. However, this does not invalidate the imminent introduction of volatility options (and volatility futures) in various exchanges. Volatility options may be proved to be very useful for volatility trading and for hedging other types of options, e.g., exotic options. The liquidity and the transaction costs will be critical factors for the success of this emerging new market, as well. In terms of the impact of model error to the hedging and pricing performance of volatility options, combinations that use simpler models such as Black-Scholes and Whaley’s seem to perform equally well with the benchmark combination (Grünbichler-Longstaff and Heston). This is in accordance with the results from previous studies in the model error (standard options) literature that found that increasing the complexity of the option pricing model does not necessarily improve its pricing and hedging performance (see e.g., Bakshi et al., 1997; Dumas et al., 1998).

The findings in Section 4 have at least five implications for the use of volatility options and their pricing models. First, volatility options are not better hedging vehicles than plain-vanilla options for the purposes of hedging standard options. Second, in the case that an investor chooses the volatility options as hedging instruments, these should be used to hedge at-the-money and out-of-the-money rather than in-the-money target options. This feature may encourage the use of volatility options for hedging purposes given that most of the options trading activity is concentrated on at-the-money options. Third, the hedging performance of volatility options increases as their rebalancing frequency increases. Fourth, the roll-over strategy with volatility options should be preferred since it performs better than the no-roll-over strategy. Finally, despite the fact that Whaley’s model is the worst mis-specified model within our simulation framework, it can be reliably used to hedge standard options with volatility options, and to price volatility options.

Finally, in Section 5, I estimated and compared diffusion and jump-diffusion stochastic volatility processes using the Maximum Likelihood Estimation (MLE). The estimation results demonstrate that the empirical fit of diffusion processes can be significantly improved by the addition of jumps. Jumps play a dominant role in implied volatility dynamics especially during periods of market stress. If jumps are conditioned
on the level of the index, model performance is further enhanced. Simulation results demonstrated that jump-diffusion processes can produce unconditional distributions that closely resemble those of the empirical data. These findings are consistent with the research on the joint dynamics of volatility and asset returns which has also suggested the presence of jumps in volatility (e.g., Duffie et al., 2000; Eraker et al., 2003; Eraker, 2004).

In the same Section I developed closed form models for pricing futures and options assuming a square root mean reverting diffusion stochastic process that allows for positive jumps in volatility. The proposed volatility option pricing models appears to have comparable properties with existing models (Grunbichler and Longstaff 1996; Detemple and Osakwe, 2000). However, it was demonstrated that incorrectly omitting jumps in volatility, may result in severe mispricing. In particular, in the case where there are upwards jumps in volatility, short (long) term volatility options are more expensive (cheaper) by as about 15% (9%). In addition, volatility calls are far less sensitive to the changes of the underlying volatility by a factor of about two.

These findings do not necessarily support criticism against the specific structural form assumed by existing volatility option pricing models. Rather, they show that pricing volatility options on a volatility index demands extreme care in modeling the underlying process. It is imperative to thoroughly understand all the issues related to pricing and hedging derivatives on volatility prior to their introduction to the market. This will ensure the smooth and successful operation of the market and the widespread adoption of volatility derivatives by investors. Alas, since no volatility options market data is yet available, we cannot use real prices to test the option pricing models.

Further future research should look at more complex specifications of the stochastic volatility process. For example, non-linear specifications of the drift/volatility structure in the spirit of Bakshi et al. (2006) could be examined in the presence of jumps in volatility. However, the non-linear specification makes the affine structure to be lost and it does not make possible the derivation of the characteristic function. This calls for an alternative econometric methodology. The jump intensity could also be allowed to depend on the level of volatility rather than being constant (see e.g., Wu, 2005). Alternative metrics (e.g., Value-at-Risk, pricing performance as soon as volatility options market data becomes available) should also be considered in order to rank the alternative implied volatility processes.

The comparative study of the hedging effectiveness of volatility options creates three strands for future research. First, the ability of volatility options to hedge exotic options (e.g., barrier options) satisfactorily should be explored. Second, the hedging effectiveness of options written on alternative measures of volatility should be investigated. For example, Brenner et al. (2001) suggest the introduction of options on straddles. Finally, the impact of the market price of volatility/jump risk to the pricing of volatility futures should be addressed.

References


<table>
<thead>
<tr>
<th>Name</th>
<th>SDE</th>
<th>Solution</th>
<th>Distribution of $V_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. GBMP</td>
<td>$dV_t = \mu V_t dt + \sigma V_t dW_t$</td>
<td>$V_t = V_0 e^{\left(\mu - \frac{1}{2} \sigma^2\right) t + \sigma W_t}$</td>
<td>$V_t \sim \text{LogN} \left(V_0 e^{\mu t}, V_0^2 e^{\mu t} \left(e^{\sigma^2 t} - 1\right)\right)$</td>
</tr>
<tr>
<td>2. MRGP</td>
<td>$dV_t = \lambda (\mu - V_t) dt + \sigma dW_t$</td>
<td>$V_t = \mu + (V_0 - \mu) e^{\lambda t} + \sigma e^{\lambda t} \int_0^t e^{-\lambda s} dW_s$</td>
<td>$V_t \sim \text{N} \left(\mu + (V_0 - \mu) e^{\lambda t}, \frac{\sigma^2}{2\lambda} + (1 - e^{-2\lambda t})\right)$</td>
</tr>
<tr>
<td>3. MRSRP</td>
<td>$dV_t = \lambda (\mu - V_t) dt + \sigma \sqrt{V_t} dW_t$</td>
<td>$V_t = V_0 e^{-\lambda t} + \mu (1 - e^{-\lambda t}) + \sigma \int_0^t e^{\lambda s} dW_s$</td>
<td>$V_t \sim \chi^2 \left(2c V_t; 2q + 2, 2u\right)$</td>
</tr>
<tr>
<td>4. MRLP</td>
<td>$d \ln V_t = \lambda (\mu - \ln V_t) dt + \sigma dW_t$</td>
<td>$V_t = V_0 e^{\lambda t} + \mu (1 - e^{-\lambda t}) + \sigma \int_0^t e^{\lambda s} dW_s$</td>
<td>$V_t \sim \text{LogN} \left(\exp\left[e^{\lambda \ln V_0 + (1 - e^{\lambda t})} + \frac{\sigma^2}{2\lambda} (1 - e^{-2\lambda t})\right], \exp\left[2e^{\lambda \ln V_0 + 2(1 - e^{\lambda t})} + \frac{\sigma^2}{2\lambda} (1 - e^{-2\lambda t})\right] \times \exp\left[\frac{\sigma^2}{2\lambda} (1 - e^{-2\lambda t})\right]^{-1}\right)$</td>
</tr>
</tbody>
</table>

**Table 1**: Stochastic volatility processes, Solutions, and their Distributional properties. GBMP stands for the Geometric Brownian Motion Process, MRGP stands for the Mean-Reverting Gaussian Process, MRSRP stands for the Mean-Reverting Square Root Process and MRLP stands for the Mean-Reverting Logarithmic Process. $\mu$ is the long run mean of volatility $V_t$ for the processes 2, 3, 4, and the rate of return of volatility for the process 1, $\sigma$ is the volatility of volatility and $\lambda$ is the speed of mean-reversion.
Table 2: Alternative models of the volatility can be nested with appropriate parameter restrictions within the unrestricted model in equation (1).

<table>
<thead>
<tr>
<th>Model</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>MRGP</td>
<td>(dV_t = \lambda (\mu - V_t) dt + \sigma dZ)</td>
</tr>
<tr>
<td>MRSRP</td>
<td>(dV_t = \lambda (\mu + V_t) dt + \sigma \sqrt{V_t} dZ)</td>
</tr>
<tr>
<td>GBMP</td>
<td>(dV_t = \mu V_t dt + \sigma V_t dZ)</td>
</tr>
<tr>
<td>MRLP</td>
<td>(d \ln(V_t) = \lambda (\mu - \ln(V_t)) dt + \sigma dZ)</td>
</tr>
<tr>
<td>GARCH-Diffusion Process</td>
<td>(dV_t = \lambda (\mu - V_t) dt + \sigma V_t dZ)</td>
</tr>
<tr>
<td>3/2 Process</td>
<td>(dV_t = \lambda (\mu - V_t) dt + \sigma V_t^{3/2} dZ)</td>
</tr>
</tbody>
</table>

Table 3: reports the parameter estimates in terms of mean reversion, long run mean and volatility. The average conditional volatility implied by each process is also reported. The \(J\)-value, the degrees of freedom (\(df\)), the \(\chi^2\) critical values for a 1% significance level and the coefficients of determination \(R^2\) and \(R^*^2\), are also reported.